Mathematics test paper

Drop.

1. The values of $\theta \in (0, 2\pi)$for which $2\,sin^2\theta - 5\,sin^2\theta - 5\,sin\theta + 2 > 0$, are

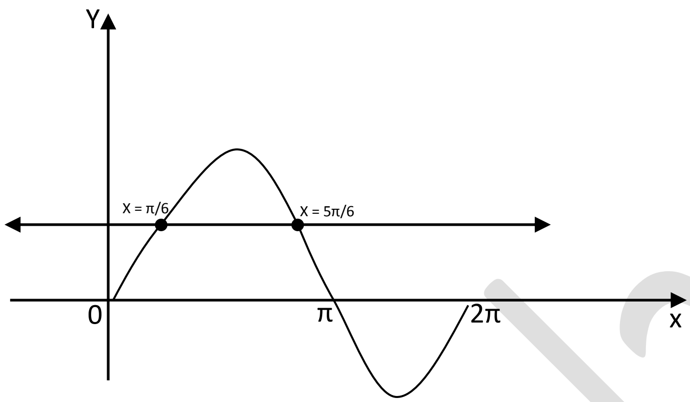
[2006-3M, -1]

(a) $\left(0,\frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (b) $\left(\frac{\pi}{8},\frac{5\pi}{6}\right)$ (c) $\left(0,\frac{\pi}{8}\right) \cup \left(\frac{\pi}{6},\frac{5\,\pi}{6}\right)$ (d) $\left(\frac{41\pi}{48},\pi\right)$

Ans. a, $2\,sin^2\theta - 5\sin\theta + 2 > 0$

$\Longrightarrow\ (sin\theta - 2)(2\sin\theta - 1) > 0$

$\Longrightarrow\ \sin\theta < \frac{1}{2}$ $[\because\ -1\ \le \sin\theta \le 1]$

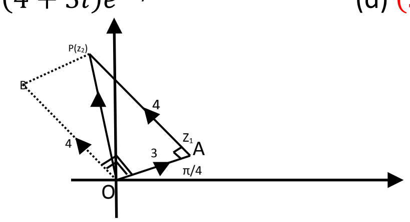


From graph, we get x ∈ $\left(0,\frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

2. A man walks a distance of 3 units form the origin towards the north-east (N 45° W) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. then the position of P in the Argand plane is [2007-3M]

(a) $3e^{i\pi/4} + 4i$ (b) $(3 - 4i)e^{i\pi/4}$ (c)$(4 + 3i)e^{i\pi/4}$ (d) $(3 + 4i)e^{\pi/4}$

Ans. d, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$



$\Longrightarrow \overrightarrow{OP} = \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{OB}}$

$\Longrightarrow \overrightarrow{OP} = 3\mathrm{e}^{\mathrm{i}\pi/4} + 4\mathrm{e}^{i(\pi/2\,+\pi/4)}$

$= 3e^{i\pi/4} + 4\mathrm{e}^{\mathrm{i}\pi/4} +.\ \mathrm{e}^{\mathrm{i}(\pi/2+\pi/4)}$

$=3e^{i\pi/4} + 4\mathrm{i}e^{i\pi/4} = e^{i\pi/4}\,(3 + 4\mathrm{i})$.

3. Let z = x + iy be complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation : z $\bar{z}^3 + \bar{z}\,z^3 = 350\ is$ [2009]

(a) 48 (b)32 (c)40 (d)80

Ans. a, Given $z\ = x + iy\ where\ x\ and\ y\ are\ integer$

Also $z\bar{z}^3 + \bar{\mathrm{z}}\mathrm{z}^3 = 350\ \Longrightarrow |z|^2\,(\bar{z}^2 + z^2) = 350$

$\Longrightarrow\quad (x^2 + y^2)(x^2 - y^2) = 175$

$\Longrightarrow\quad (x^2 + y^2)(x^2 - y^2) = 25 \times 7$ ...(i)

or $\quad (x^2 + y^2)(x^2 - y^2) = 35 \times 5$ ...(ii)

$\because$ x and y are integers,

$\therefore\quad x^2 + \mathrm{y}^2 = 25 \text{ and } \mathrm{x}^2 - \mathrm{y}^2 = 7$ [From eq (i)]

$\Longrightarrow\quad x^2 = 16 \text{ and } \mathrm{y}^2 = 9$

$\Longrightarrow$ x = $\pm$ 4 and y = ± 3

$\therefore$ Vertices of rectangle are

$(4,3), (4,-3), (-4,-3), (-4,3)$.

So, area of rectangle = 8 × 6 = 48 sq. units

Now form eq.(ii)

Or $x^2 + y^2 = 35$ and $x^2 - y^2 = 5$

$\Longrightarrow x^2 = 20$, which is not possible for any integral value of x

4. The set of all real numbers x for which $x^2 - |x+2| + x > 0, is$

(a) $(-\infty, -2) \cup (2, \infty)$ (b) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(c) $(-\infty, -1) \cup (1, \infty)$ (d) $(\sqrt{2}, \infty)$

Ans. b, For $x < -2, |x+2| = x+2$ and the inequality becomes

$x^2 + x + 2 + x > 0 \quad \Longrightarrow (x+1)^2 + 1 > 0$

Which is valid ∀ x ∈ R but x < -2

$\therefore \quad x \in (-\infty, -2)$ ...(1)

For $x \geq 2, |x+2| = x+2$ *and the inequality becomes*

$x^2 - x - 2 + x > 0 \Longrightarrow x^2 > 2 \Longrightarrow x > \sqrt{2}$ or $x < -\sqrt{2}$

i.e., $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

but $x \geq -2 \Longrightarrow x \in [-2, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ ...(2)

From (1) and (2)

$x \in (-\infty, -2) \cup [-2, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

$\Longrightarrow \quad x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

5. The quadratic equation p(x) =0 with real coefficients has purely imaginary roots. Then the equation p(p(x)) =0 has [JEE Adv. 2014]

(a) One purely imaginary root (b) all real roots

(c) Two real and two purely imaginary roots (d) neither real nor purely imaginary roots

Ans. d, Quadratic equation with real coefficients and purely imaginary roots can be considered as

$p(x) = x^2 + a = 0$ *where* $a > 0$ *and* $a \in R$

The $p[p(x)] = 0 \Longrightarrow (x^2 + a)^2 + \text{a} = 0$

$\Longrightarrow \quad x^4 + 2\text{a}x^2 + (a^2 + a) = 0$

$\Longrightarrow \quad x^2 = \frac{-2a \pm \sqrt{4a^2 - 4a^2 - 4a}}{2}$

$\Longrightarrow \quad x^2 = -\text{a} \pm \sqrt{a}\, i$

$\Longrightarrow \quad x = \sqrt{-a \pm \sqrt{a} i} = \text{a} \pm \text{i}\beta$ where a, $\beta \neq 0$

$\therefore \quad p[p(x)] = 0$ has complex roots which are neither purely real nor purely imaginary.

6. Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearings the same number and moreover the card number 1 is always placed in envelope numbered 2. Then the number of ways it can be done is [JEE Adv. 2014]

(a) 264 (b) 265 (c) 53 (d) 67

Ans. c, ∵ Card numbered 2 is placed in envelope numbered 2, we can consider two cases.

Case I: Card numbered 2 is placed in envelope numbered 1. Then it is dearrangement of 4 objects, which can be done in $4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$ ways

Case II: Card numbered 2 is not placed in envelope numbered 1.

Then it is dearrangement of 5 objects, which can be done in $5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44\ ways$

$\therefore$ Total ways $= 44 + 9 = 53$

7. How many different nine digit numbers can be formed from the number 23355888 by rearranging its digits so that the odd digits occupy even positions? [2000S]

   (a) 16 (b) 36 (c) 60 (d) 180

Ans. c, $X - X - X - X - X$. The four digits 3,3,5,5 can be arranged at$(-)$Places in $\frac{4!}{2!2!}$ =6 ways.

The five digits 2,2,8,8,8 can be arranged at

(X) places in $\frac{5!}{2!3!} = 10$ ways.

Total no. of arrangements $= 6 \times 10 = 60$ ways.

8. The sum of the first n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \cdots$ is $\frac{n(n+1)^2}{2}$ when n is even. When n is odd the sum is [2004]

   (a) $\left[\frac{n(n+1)}{2}\right]^2$ (b) $\frac{n^2(n+1)}{2}$ (c) $\frac{n(n+1)^2}{4}$ (d) $\frac{3n(n+1)}{2}$

Ans. b, If n is odd, the required sum is

$1^2 + 2.2^2\ 3^2 + 2.4^2 + \cdots + 2.(n-1)^2$

$= \frac{(n-1)(n-1+1)^2}{2} + n^2$

$[\because (n-1)\ is\ even\ \therefore using\ given\ formula\ for\ the\ sum\ of\ (n-1) terms.]$

$= \left(\frac{n-1}{2} + 1\right)n^2 = \frac{n^2(n+1)}{2}$

9. If $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$ *where* a, b, c are in A.P and $|a| < 1, |b| < 1, |c| < 1$ then x, y, z are in [2005]

   (a) G.P. (b) A.P.

   (c) Arithmetic- Geometric Progression (d) H.P.

Ans. d, $x = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ $\quad$ $a = 1 - \frac{1}{x}$

$y = \sum_{n=0}^{\infty} b^n = \frac{1}{1-b}$ $\quad$ $b = 1 - \frac{1}{y}$

$z = \sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$ $\quad$ $c = 1 - \frac{1}{z}$

a, b, c are in A.P. or 2b = a + c

$2\left(1 - \frac{1}{y}\right) = 1 - \frac{1}{x} + 1 - \frac{1}{y}$

$\frac{2}{y} = \frac{1}{x} + \frac{1}{z} \implies x, y, z\ are\ in\ H.P$

10. The sum to infinite term of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \cdots is$ [2009]

   (a) 3 (b) 4 (c) 6 (d) 2

Ans. a, we have

$S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \cdots\ \infty$ ...(1)

Multiplying both sides by $\frac{1}{3}$ we get

$\frac{1}{3}S = \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \cdots \infty$ ...(2)

$subtracting\ eqn. (2) from\ eqn. (1) we\ get$

$\frac{2}{3}S = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \cdots \infty$

$\implies \frac{2}{3}S = \frac{4}{3} + \frac{4}{3^2} + \frac{4}{3^4} + \cdots \infty$

$\implies \frac{2}{3}S = \frac{\frac{4}{3}}{1-\frac{1}{3}} = \frac{4}{3} \times \frac{3}{2} \implies S = 3$

11. The points $(-a, -b), (0,0), (a, b)$ and (a^2, ab) are : [1979]

   (a) Collinear (b) vertices of a parallelogram

(c) Vertices of a rectangle (d) None of these

Ans. a, the given points are $A(-a, -b), B(0,0), C(a, b)$ and $D\ (a^2, ab)$.

Slope of AB $= \frac{b}{a} =$ slope of BC $=$ slope of BD

$\therefore$ A, B, C, D are collinear.

12. The straight lines x + y =0, 3x + y – 4 = 0, x + 3y – 4 =0 form a triangle which is [1983 -1M]

(a) Isosceles (b) equilateral (c) right angled (d) none of these

Ans. a, solving the given equations of lines pairwise, we get the vertices of Δ as

$A\ (-2,2) B(2,-2) C(1,1)$

Then AB $= \sqrt{16+16} = 4\sqrt{2}$

BC $= \sqrt{1+9} = \sqrt{10}$

CA $= \sqrt{9+1} = \sqrt{10}$ $\therefore$ Δ is isosceles.

13. If x satisfies $|x-1| + |x-2| + |x-3| \geq 6$, then,

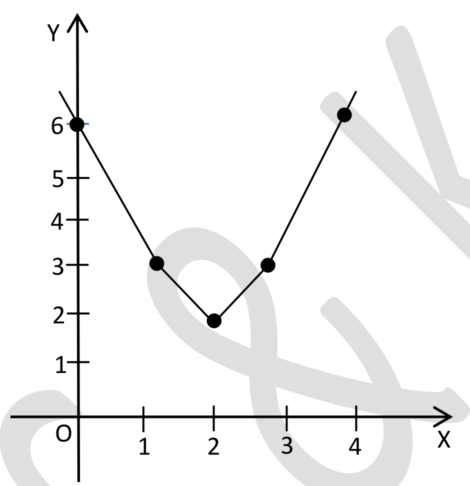
(a) $0 \leq x \leq 4$ (b) $x \leq -2$ or $x \geq 4$ (c) $x \leq 0$ or $x \geq 4$ (d) none of these

Ans. c, $|x-1| + |x-2| + |x-3|) \geq 6$

Consider $f(x) = |x-1| + |x-2| + |x-3|$

$$f(x) = \begin{cases} 6-3x, & x < 1 \\ 4-x, & 1 \leq x < 2 \\ x, & 2 \leq x < 3 \\ 3x-6, & x \geq 3 \end{cases}$$

NOTE THIS STEP :



Graph of $f(x)$ shows $f(x) \geq 6$ for $x \leq 0$ or $x \geq 4$

14. Let $f: R \to R$ be any function. Define g: R → R by g(x) = $|f(x)|$ for all x. then g is [2000S]

(a) Onto if f is onto

(b) One-one if f is one-one.

(c) Continuous if f is continuous

(d) Differentiable if f is differentiable.

Ans. c, Let h $(x) = |x|$ then

g(x) $= |f(x)| =$ h$(f(x))$

Since composition of two continuous functions is continuous, therefore g is continuous if f is continuous.

15. Let $f(x) = \frac{ax}{x+1}, x \neq -1$. Then, for what value of a is $f(f(x)) = x$? [2001S]

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 1 (d) -1

Ans. d, $f(x) = \frac{ax}{x+1}, x \neq -1$

$f(f(x)) = x \implies \frac{a\left(\frac{ax}{x+1}\right)}{\frac{ax}{x+1}+1} = x \implies \frac{a^2 x}{(a+1)x+1} = x$

$\implies (a+1)x^2 + (1-a^2)x = 0$ ...(1)

$\implies a+1 = 0$ and $1 - a^2 = 0 \implies a = -1$

16. If the functions $f(x)$ and $g(x)$ are defined on R → R such that $f(x) = \begin{cases} 0, & x \in rational \\ x, & x \in irrational \end{cases}$; $g(x) = \begin{cases} 0, & x \in irrational \\ x, & x \in rational \end{cases}$ then $(f-g)(x)$ is [2005S]

(a) One-one & onto

(b) Neither one-one nor onto

(c) One-one but not onto

(d) Onto but not one-one

Ans. a, We are given that

$f : R \to R$ such that $f(x) = \begin{cases} 0, & x \in rational \\ x, & x \in irrational \end{cases}$

$g : R \to R \text{ such that } g(x) = \begin{cases} 0, & x \in irrational \\ x, & x \in ratinal \end{cases}$

$\therefore (f-g)$: R → R such that

$(f-g)(x) = \begin{cases} -x, & if\ x \in rational \\ x, & if\ x \in irrational \end{cases}$

Since $f - g : R \to R$ for any x there is only one value of $(f(x) - g(x))$ whethere x is rational or irrational. Moreover as $x \in R, f(x) - g(x)$ also belongs to R. therefore, $(f-g)$ is one-one onto.

17. If $f(a) = 2, f'(a) = 1, g(a) = -1, g'(a) = 2, then\ the\ value\ of \lim_{x\to a} \frac{g(x)f(a)-g(a)f(x)}{x-a}$ is [1983-1M]

(a) -5 (b) $\frac{1}{5}$ (c) 5 (d) none of these

Ans. c, $\lim_{x\to a} \frac{g(x)\, f(a) - g(a) f(x)}{x-a}$

$= \lim_{h\to 0} \frac{g(a+h)f(a) - g(a)f(a+h)}{h}$ $[For\ x = a+h\ ]$

$= \lim_{h\to 0} \frac{g(a+h)f(a) - g(a)f(a) + g(a)\, f(a) - g(a)\, f(a+h)}{h}$

$= \lim_{h\to 0} f(a)\left[\frac{g(a+h)-g(a)}{h}\right] - \lim_{h\to 0} g(a)\left[\frac{f(a+h)-f(a)}{h}\right]$

$= f(a)\, g'(a) - g(a)\, f'(a) = 2\times2-(-1)\times1 = 5$

18. $\lim_{x\to 0} \frac{x \tan 2x - 2x \tan x}{(1-\cos 2x)^2}$ is [1999-2M]

(a) 2 (b) -2 (c) ½ (d)-1/2

Ans. c, $\lim_{x\to 0} \frac{x \tan 2x - 2x \tan x}{(1-\cos 2\,x)^2}$

$= \lim_{x\to 0} \frac{x\left\{2x + \frac{8x^3}{3} + \frac{64x^5}{15} + \cdots\right\} - 2x\left\{x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right\}}{4 \sin^4 x}$

$= \lim_{x\to 0} \frac{x^4\{\frac{8}{3} - \frac{2}{3} + terms\ containing\ higher\ positive\ powers\ of\ x}{4 \sin^4 x}$

$= \frac{1}{4} . 2 = \frac{1}{2}$

19. Let $f(x) = \begin{cases} x^2 \left|\cos\frac{\pi}{x}\right|, & x \neq 0 \\ 0, & x = 0 \end{cases}, x \in R\ then\ f\ is$

(a) Differentiable both at x = 0 and at x = 2

(b) Differentiable at x = 0 but not differentiable at x = 2

(c) Not differentiable at x =0 but differentiable at x = 2

(d) Differentiable neither at x=0 not at x = 2

Ans. b, we have $f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right|}{h} = \lim_{h\to 0}\left|\cos\frac{\pi}{h}\right|$

$= 0 \times$ some finite value $= 0$

Also, $f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right|}{m}$

$= \lim_{h\to 0} h\left|\cos\frac{\pi}{h}\right| = 0 \times$ some finite value $= 0$

$\because \quad f'(0^+) = f'(0^-) \implies f$ is differentiable at x =0

Now $\quad f'(2^+) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$

$= \quad \lim_{h\to 0}\frac{(2+h)^2\left|\cos\frac{\pi}{2+h}\right|-4\left|\cos\frac{\pi}{2}\right|}{h}$

$= \quad \lim_{h\to 0}\frac{(2+)h^2\left(\cos\frac{\pi}{2+h}\right)}{h}$

$= \quad \lim_{h\to 0}\left(\frac{(2+h)^2}{h}\right)\sin\left(\frac{\pi}{2}-\frac{\pi}{2+h}\right)$

$= \quad \lim_{h\to 0}\frac{(2+)h^2}{h}\sin\left(\frac{\pi h}{2(2+)h}\right)$

$= \quad \lim_{h\to 0}\frac{(2+h)^2}{h}\times\frac{\sin\left(\frac{\pi h}{2(2+h)}\right)}{\left(\frac{\pi h}{2(2+h)}\right)}\times\frac{\pi h}{2(2+h)} = \pi$

Also $f'(2^-) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

$= \quad \lim_{h\to 0}\frac{(2-h)^2\left|\cos\left(\frac{\pi}{2-h}\right)\right|-0}{-h}$

$= \quad \lim_{h\to 0}\frac{-(2-h)^2\cos\left(\frac{\pi}{2-h}\right)}{-h}$

$= \quad \lim_{h\to 0}\frac{(2-h)^2\sin\left(\frac{\pi}{2}-\frac{\pi}{2-h}\right)}{h}$

$= \quad \lim_{h\to 0}\frac{(2-h)^2}{h}\times\frac{\sin\left(\frac{-\pi h}{2(2-h)}\right)}{\left(\frac{-\pi h}{2(2-h)}\right)}\times\left(\frac{-\pi h}{2(2-h)}\right) = -\pi$

As $\quad f'(2^+) \neq f'(2^-) \implies f$ is not differentiable at x = 2.

20. Let y be an implicit function of x defined by $x^{2x} - 2x^x\cot y - 1 = 0$. Then y' (1) equals [2009]

(a) 1 (b) log 2 (c) –log 2 (d) -1

Ans. d, $x^{2x} - 2x^x\cot y - 1 = 0$

$\implies 2\cot y = x^x - x^{-x} \implies 2\cot y = u - \frac{1}{u}$ where $u = x^x$

Differentiating both sides with respect to x, we get

$\implies -2cosec^2\, y\frac{dy}{dx} = \left(1+\frac{1}{u^2}\right)\frac{du}{dx}$

Where $u = x^x \implies \log u = x\log x$

$\implies \quad \frac{1}{u}\frac{du}{dx} = 1+\log x \implies \frac{dy}{dx} = x^x\,(1+\log x)$

$\therefore$ we get $-2\,cosec^2\, y\frac{dy}{dx} = (1+x^{-2x})\,x^x\,(1+\log x)$

$\implies \quad \frac{dy}{dx} = \frac{(x^x+x^{-x})(1+\log x)}{-2(1+cot^2 y)}$ ...(i)

Now when x = 1, $x^{2x} - 2x^x\cot y - 1 = 0, gives$

$1 - 2\cot y - 1 = 0 \implies \quad \cot y = 0$

$\therefore \quad form\ equation\ (i), at\ x = 1\ and \cot y = 0, we\ get$

$y'(1) = \frac{(1+1)(1+0)}{-2(1+0)}$ = -1

21. $\frac{d^2x}{dy^2}$ *equals* : [2011]

(a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (c) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

Ans. c, $\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right)\frac{dx}{dy}$

$= \frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy} = -\frac{1}{\left(\frac{dy}{dx}\right)^2}.\frac{d^2y}{dx^2}.\frac{1}{\frac{dy}{dx}} = -\frac{1}{\left(\frac{dy}{dx}\right)^3}\frac{d^2y}{dx}$

22. If x = -1 and x = 2 are extreme points of $f(x) = a\log|x| + \beta x^2 + x$ then [JEE M 2014]

(a) $a = 2, \beta = -\frac{1}{2}$ (b)$aq = 2, \beta = \frac{1}{2}$ (c)a = -6, $\beta = \frac{1}{2}$ (d) $a = -6, \beta = -\frac{1}{2}$

Ans. a, let $f(x) = a\log|x| + \beta x^2 + x$

Differentiating both sides,

$f'(x) = \frac{a}{x} + 2\beta x + 1$

Since x = -1 and x = 2 are extreme points therefore $f'(x) = 0$ at these points.

Put x = -1 and x = 2 in $f'(x)$, we get

$-a - 2\beta + 1 = 0 \Rightarrow a + 2\beta = 1$ ...(i)

$\frac{a}{2} + 4\beta + 1 = 0 \Rightarrow a + 8\beta = -2$ ...(ii)

On solving (i) and (ii), we get

$6\beta = -3 \Rightarrow \beta = -\frac{1}{2}$ $\therefore a = 2$

23. If the angles of a triangle are in the ratio 4 : 1:1, then the ratio of the longest side to the perimeter is [2003S]

(a)$\sqrt{3}:(2+\sqrt{3})$ (b)1: 6 (c) 1: 2 + $\sqrt{3}$ (d) 2: 3

Ans. a, Given that $A : B : C = 4 : 1 : 1$

Let A = 4x, $B = x, C = x$

But A + B + C = 180°

$\Rightarrow$ 4x + x + x = 180° $\Rightarrow$ x = 30°

$\therefore$ $a = 120°, b = 30°, C = 30°$

By sine low, $\frac{a}{\sin 120°} = \frac{b}{\sin 30°} = \frac{c}{\sin 30°}$

$\Rightarrow$ $\frac{a}{\sqrt{3}\frac{}{2}} = \frac{b}{\frac{1}{2}} = \frac{c}{\frac{1}{2}} \Rightarrow a : b : c = \sqrt{3} : 1 : 1$

$\therefore$ Ration of longest side to the perimeter

$= \sqrt{3} : 1 + 1 + \sqrt{3} = \sqrt{3} : 2 + \sqrt{3}$

24. Let ABCD be a quadrilateral with area 18, with side AB parallel to the side CD and 2AB = CD. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral ABCD touching all the sides, then its radius is [2007-3M]

(a) 3 (b) 2 (c) $\frac{3}{2}$ (d) 1

Ans. b, Given AB || CD, CD = 2AB let AB = a then CD = 2a

Let radius of circle be r. be let circle touches AB at P, BC at Q, AD at CD at S.

Then AR = AP =r, BP = BQ = a – r

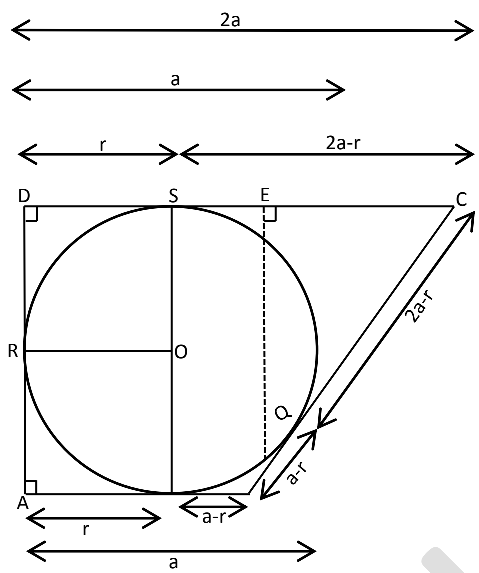
DR = DS = r and CQ = CS = 2a – r In ΔBEC

$BC^2 = BE^2 + EC^2 \Rightarrow (a - r + 2a - r)^2 = (2r)^2 + (a)^2$

$\Rightarrow$ $9a^2 + 4r^2 - 12\,ar = 4r^2 + a^2$

$\Rightarrow \qquad a = \frac{3}{2}r \qquad \dots (1)$

Also Ar (quad. ABCD) = 18



$\Rightarrow \quad a \times 2r + \frac{1}{2} \times a \times 2r = 18$

$\Rightarrow \quad$ ar =6 $\Rightarrow \frac{3r^2}{2} = 6 \qquad (using\ equation\ (1))$

$\Rightarrow \quad r^2 = 4 \ \Rightarrow \quad r = 2$

25. If the normal to the curve y $= f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ *with* the positive x-axis, then $f'(3) =$ [2000S]

(a) -1 (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 1

Ans. d, slope of tangent y $= f(x) is \frac{dy}{dx} = f'(x)_{(3,4)}$

Therefore, slope of normal = $-\frac{1}{f'(x)_{3,4}} = -\frac{1}{f'(3)}$

But $-\frac{1}{f'(3)} = \tan\left(\frac{3\pi}{4}\right)$ $(given)$

Or $\quad -\frac{1}{f'(3)} = \tan\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = -1 \ \Rightarrow f'(3) = 1$

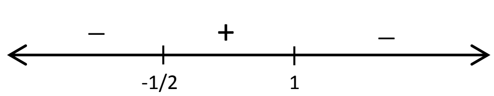
26. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is [2001S]

(a) Increasing on [-1/2, 1] (b) decreasing on R (c) increasing on R (d) decreasing on [-1/2,1]

Ans. a, $f(x) = xe^{x(1-x)}$

$\Rightarrow f'(x) = e^{x(1-x)} + (1-2x)xe^{x(1-x)}$

$= e^{x(1-x)}(2x^2 - x - 1) = -e^{(1-x)}(2x+1)(x-1)$



$\therefore \quad f(x)$ is increasing on $[-1/2, \ 1]$

27. If $f(x) = x^3 + bx^2 + cx + d$ *and* $0 < b^2 < c$, *then in* $(-\infty, \infty)$ [2004S]

(a) $f(x)$ is a strictly increasing function

(b) $f(x)$ has a local maxima

(c) $f(x)$ is a strictly decreasing function

(d) $f(x)$ is bounded

Ans. a, $f(x) = x^3 + bx^2 + cx + d, 0 < b^2 < c$

$f'(x) = 3x^2 + 2bx + c$

Discriminant $= 4b^2 - 12c = 4(b^2 - 3c) < 0$

$\therefore$ $f'(x) > 0\ \forall\ x \in R$

$\Rightarrow f(x) is\ strictly\ increasing\ \ \forall\ x\ \in R$

28. In [0,1]Lagranges mean value theorem in NOT applicable to [2003S]

(a) $f(x) = \begin{cases} \frac{1}{2} - x\ x < \frac{1}{2} & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & x \geq \frac{1}{2} \end{cases}$

(b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(c) $f(x) = x|x|$

(d) $f(x) = |x|$

Ans. a, There is only one function in option(a) whose critical point $\frac{1}{2} \in (0, 1)$ for the rest of the parts critical point $0\ \notin (0, 1)$. It can be easily seen that function in option (b), (c) and (d) are continuous on $[0, 1]$ and differentiable in $(0, 1)$.

Now for $f(x) = \begin{cases} \left(\frac{1}{2} - x\right), x < 1/2 \\ \left(\frac{1}{2} - x\right)^2, x\ \geq 1/2 \end{cases}$

Here $f'\left(\frac{1^-}{2}\right) = \ -1$ and $f'(1/2^+) = -2\left(\frac{1}{2} - \frac{1}{2}\right) = 0$

$\therefore$ $f'\left(\frac{1^-}{2}\right) \neq f'(1/2^+)$

$\therefore$ f is not differentiable at $1/2 \in (0, 1)$

$\therefore$ LMV is not applicable for this function in $[0, 1]$

29. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval [1997-2M]

(a) Both $f(x)$ and $g(x)$ are increasing functions

(b) Both $f(x)$ and $g(x)$ are decreasing functions

(c) $f(x)$ is an increasing function

(d) g(x) is an increasing function

Ans. c, We have $f(x) = \frac{x}{\sin x}, 0 <\ x\ \leq 1$

$\Rightarrow$ $f'(x) = \frac{\sin x - \cos x}{sin^2 x}$

Where $sin^2 x\ is\ always + ve, when$ $0 < x\ \leq 1$. But to check Nr., we again let

$h(x) = \sin x - x\cos x$

$\Rightarrow$ $h'(x) =\ x\sin x > 0\ for\ 0 < x\ \leq 1 \Rightarrow h(x) is\ increasing$

$\Rightarrow$ $h(0) < h(x), when\ 0 < x \leq 1$

$\Rightarrow$ $0 < \sin x - x\cos x > 0$, when $0 < x \leq 1$

$\Rightarrow$ $\sin x - x\cos x > 0$, when $0 < x \leq 1$

$\Rightarrow$ $f'(x) > 0, x\ \in (0, 1]$

$\Rightarrow$ $f(x)$ is increasing on $(0, 1]$

Again $g(x) = \frac{x}{\tan x}$

$\Rightarrow$ $g'(x) = \frac{\tan x - x\, sec^2 x}{tan^2 x}$, when $0 < x \leq 1$

Here $tan^2 x > 0$ *but to check Nr. we consider*

$p(x) = \tan x - x\, sec^2 x$

$p'(x) = sec^2 x - sec^2 x - x.2 \sec x . \sec x \tan x$

$\Rightarrow$ $p'(x) = -2x\, sec^2 x \tan < 0$ $for \; < x \leq 1$

$\Rightarrow$ $p(x)$ is decreasing, when $0 < x \leq 1$

$\Rightarrow$ $P(0) > P(x) \Rightarrow 0 > \tan x - x\, sec^2 x$

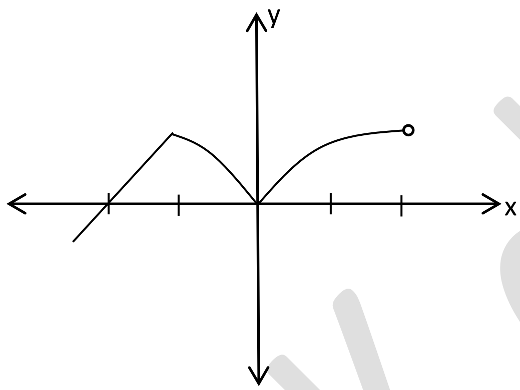
$\therefore$ $g'(x) < 0$

Hence g(x) is decreasing when $0 < x \leq 1$.

30. the total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3 & , -3 < x \leq -1 \\ x^{2/3} & , -1 < x < 2 \end{cases}$ [2008]

(a)0 (b) 1 (c)2 (d)3

Ans. c,



The given function is

$f(x) = \begin{cases} (2+x)^3, & -3 < x \leq -1 \\ x^{2/3}, & -1 < x - 2 \end{cases}$

The graph of y = f(x) is as shown in the figure. From graph, clearly, there is one local maximum ($at\ x = -1$) and one local minima ($at\ x = 0$)

$\therefore$ total number of local maxima or minima = 2.

Answer key

Ques.	1	2	3	4	5	6	7	8	9	10
Ans.	a	d	a	b	d	c	c	b	d	a
Ques.	11	12	13	14	15	16	17	18	19	20
Ans.	a	a	c	a	d	a	c	c	b	d
Ques.	21	22	23	24	25	26	27	28	29	30
Ans.	c	a	a	b	d	a	a	a	c	c